A single exponential FPT algorithm for the K_4-minor cover problem

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Joint work with
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July 4, 2012

Parameterized K_4-Minor Cover

Given a graph $G = (V, E)$ and an integer k as parameter,

- at most k vertices $S \subseteq V$ s.t $G[V \setminus S]$ is K_4-minor free ?

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(a.k.a. Parameterized Treewidth-two Vertex Deletion)

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Observations :

1. Vertex Cover $\equiv$ K_2-Minor Cover
   $\equiv$ Treewidth-zero Vertex Deletion
2. Feedback Vertex Set
   $\equiv$ K_3-Minor Cover
   $\equiv$ Treewidth-one Vertex Deletion

More generally,

How fast can we solve Treewidth-t Vertex Deletion ?

Known results (*when we submitted)

1. Parameterized K_4-Minor Cover is FPT (by the Roberston and Seymour' graph minor theorem or by Courcelle's theorem)
2. Best algorithm runs in $2^{O(k \log k)} \cdot n^{O(1)}$ [Fomin et al.'11]
3. $2^{O(k)} \cdot n^{O(1)}$-algorithm when for $t = 0, 1$.
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Our result

There exists an algorithm that solves the Parameterized K_4-Minor Cover problem in time $2^{O(k)} \cdot n^{O(1)}$.

Known results (*now, a few months later...)

1. Treewidth-t Vertex Deletion in $2^{O(k)} \cdot n \log n^2$ [Fomin et al.'12], in $2^{O(k)} \cdot n^2$ [Kim et al.'12]
2. Polynomial kernel [Fomin et al.'12]

Iterative compression

allows us to focus on DISJOINT-K_4-MINOR COVER

- Given $G = (V, E)$, a K_4-MINOR COVER S of size $k + 1$
- Compute (if it exists) a K_4-MINOR COVER S' of size k such that $S \cap S' = \emptyset$

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Our algorithm for DISJOINT-K_4-MINOR COVER can be viewed as a generalization of [Chen et al.08] for DISJOINT-FVS.

DISJOINT-FEEDBACK VERTEX SET (DISJOINT-FVS)

- Given $G = (V, E)$, a feedback vertex set S of size $k + 1$
- Compute (if it exists) a feedback vertex set S' size k such that $S \cap S' = \emptyset$.

Disjoint-Feedback Vertex Set (Disjoint-FVS)

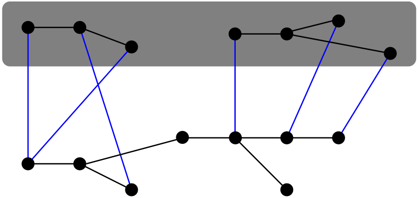
- Given $G = (V, E)$, a feedback vertex set S of size $k + 1$
- Compute (if it exists) a feedback vertex set S' size k such that $S \cap S' = \emptyset$.

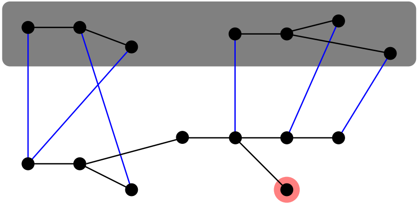
[Chen et al.08] We use

- branching and reduction rules
- a measure function to analyze the time complexity

$$\boxed{\mu = k + \#cc(G[S])}$$

Skip example





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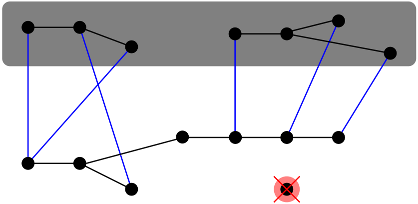

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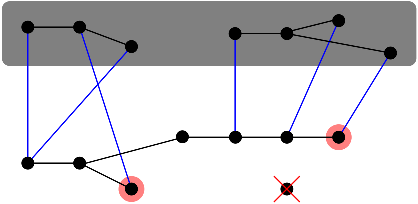

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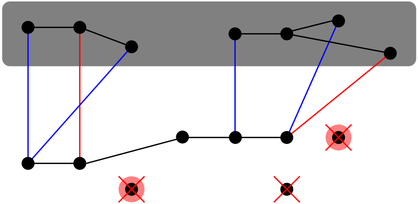

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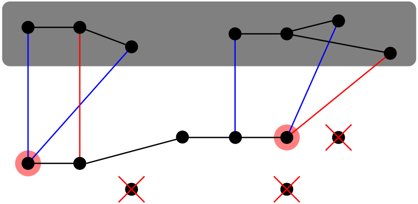

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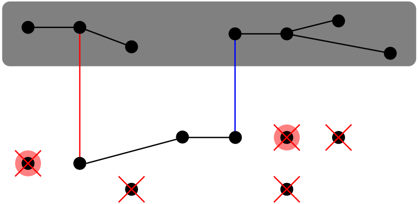

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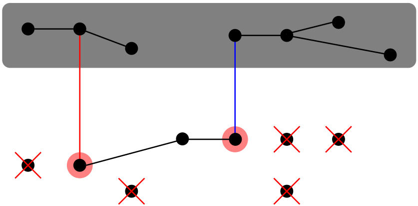

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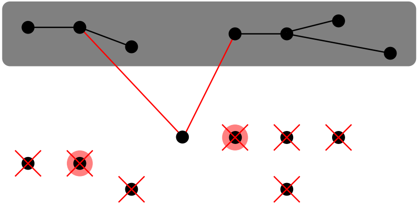

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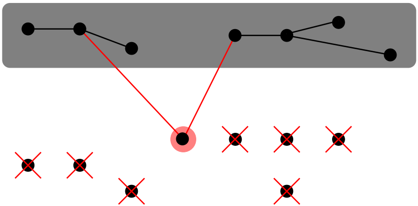

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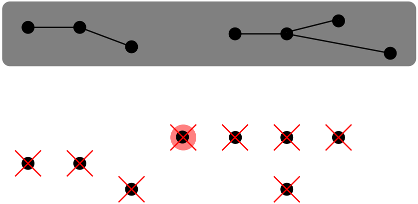
- $(G - \{x\}, S, k - 1)$ $\Rightarrow$ μ decreases



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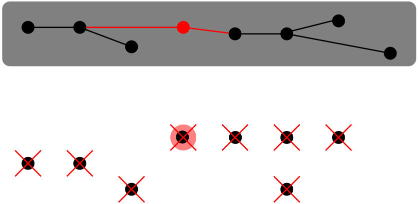

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- $(G, S \cup \{x\}, k)$ $\Rightarrow \mu$ decreases

Ingredients of [Chen et al.08]

- Branching rules AND appropriate measure function μ.
- Reduction rules to bound the branching degree.
- An appropriate tree-like structure to process $G - S$.
- In the search tree, leaf instances are not hard.

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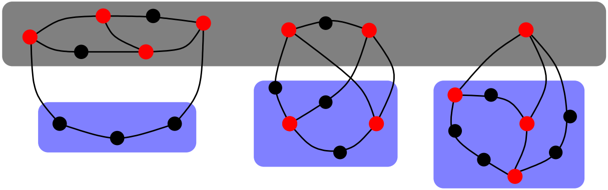
For the Disjoint-K_4-Minor Cover we have

- adapted the branching rules and introduce a new measure μ
- adapted the reduction rules (extended bypassing + chandelier + trivial)
- extended SP-decomposition for treewidth-2 graphs.
- in the search tree, a leaf instance is Vertex Cover on circle graphs (polytime).

Branching Rules (1)

Let (G, S, k) be an instance of Disjoint-K_4-Minor Cover.



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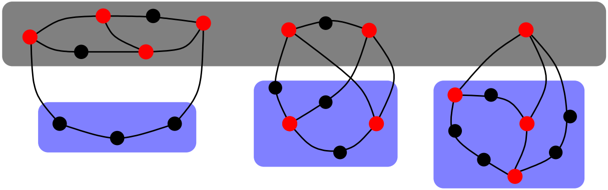

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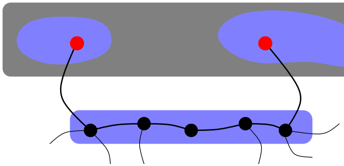

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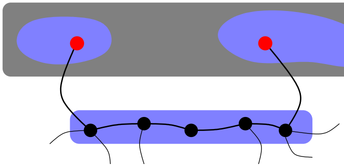

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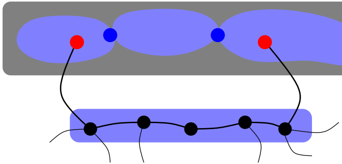

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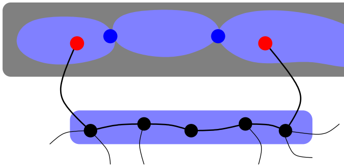

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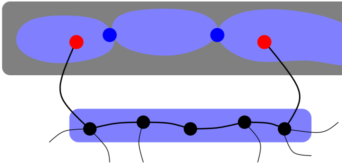

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- c_1 depends on the maximum size of the sets X on which the branching rules is applied

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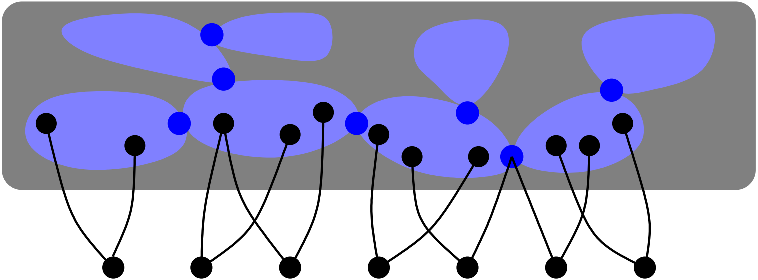

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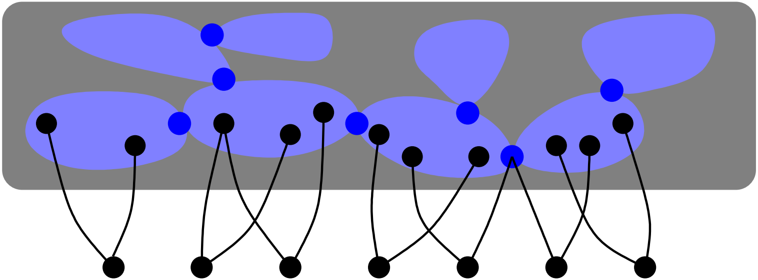

independent instance can be solved in poly-time

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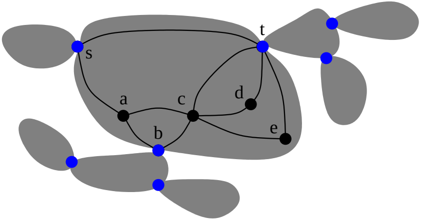
extended-SP decomposition



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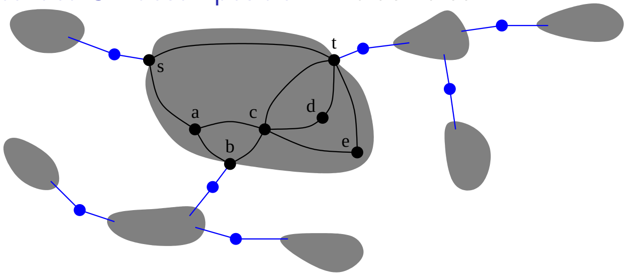
extended-SP decomposition = block tree



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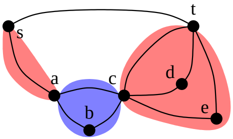
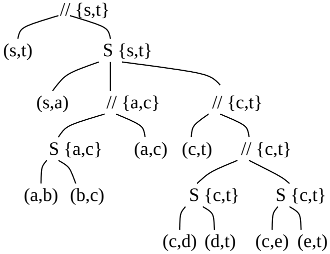
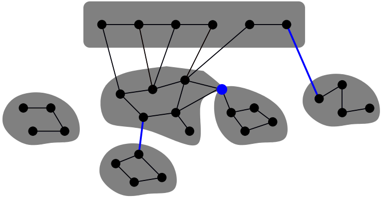
extended-SP decomposition = block tree + SP-tree on every block





Reduction Rules: trivial



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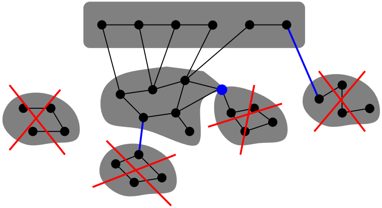

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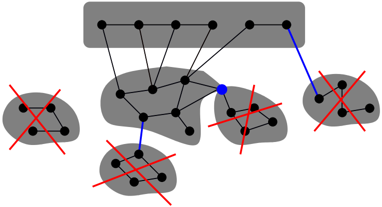
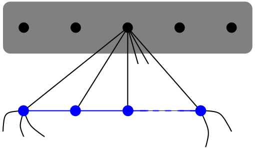


Reduction rule: Components NOT participating any K_4-subdivision is removed.

Reduction rule: Bypass degree-2 vertices and remove multiple edges.

Reduction Rules: Chandelier



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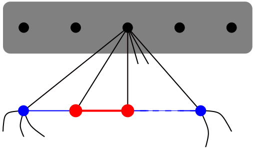

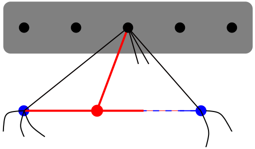

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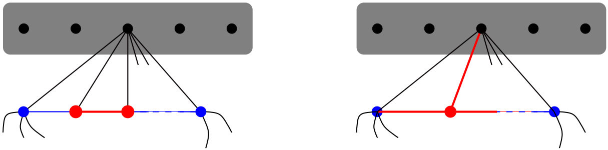

Reduction Rules: extended bypass-1

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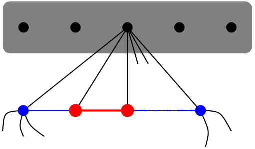
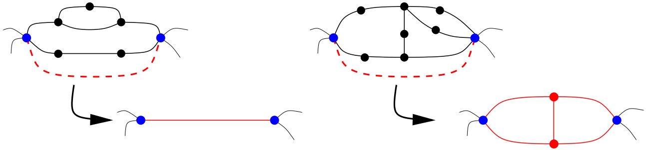


Reduction Rules: extended bypass-1

when connected X s.t. $X \cap S = \emptyset$ has a separator of size 2

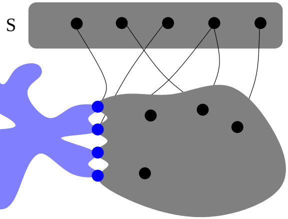


Reduction Rules: extended bypass-2

Let (G, S, k) be an instance of Disjoint-K_4-Minor Cover

Disjoint Protrusion Rule: Let X be a t-protrusion of G such that

$$X \cap S = \emptyset \text{ and } |X| > \gamma(t)$$



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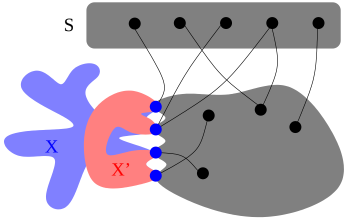

Then, replace X with a t-protrusion X' of smaller size.

Algorithm outline

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3. Solve each independent instance in polytime

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Thank you