On Minimum Sum of Radii and Diameters Clustering

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- **Goal:** partition the points of V into at most k clusters $V_1, V_2, \ldots, V_k$.
- **Objective:** minimize $\sum_{i=1}^{k} \text{rad}(V_i)$ in MSR, minimize $\sum_{i=1}^{k} \text{diam}(V_i)$ in MSD.
- **Radius and Diameter:** $\text{rad}(V_i) = \min_{u \in V_i} \max_{v \in V_i} d(u, v)$, $\text{diam}(V_i) = \max_{u,v \in V_i} d(u, v)$

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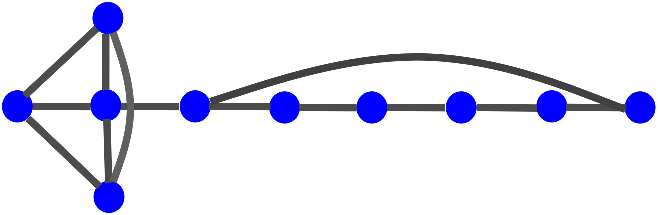

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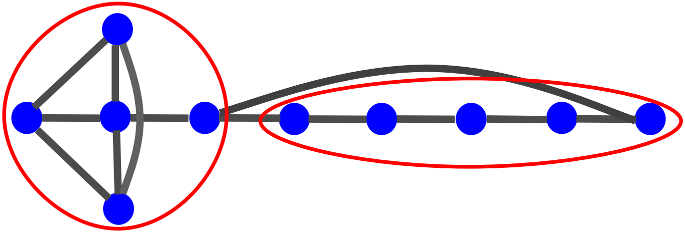

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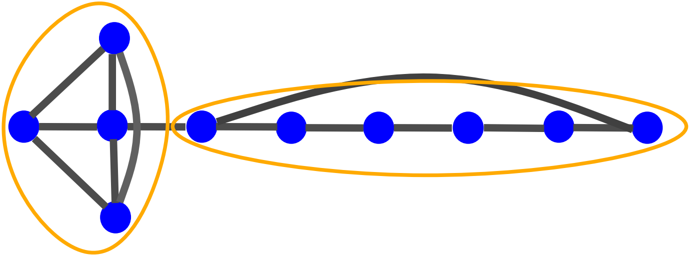

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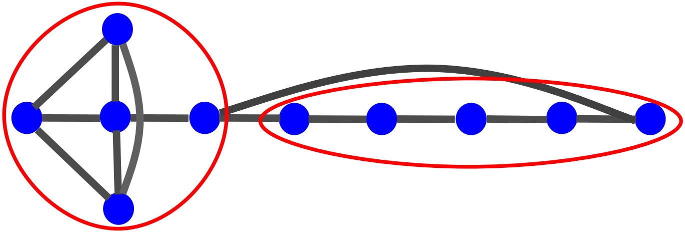

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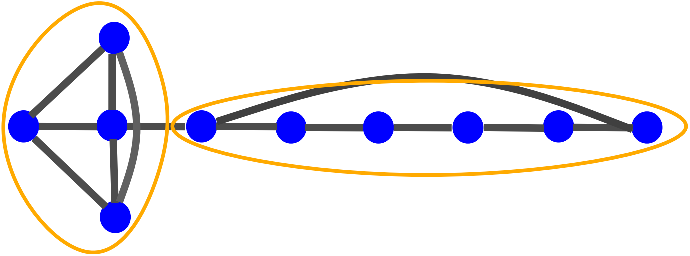

Motivations

Clustering

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Communication Networks

Location of base stations in a wireless data network.

MSR and MSD Previous Works: the general case

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- exact algorithm for MSR in time $n^{O(\log n \log \Delta)}$ where Δ is the ratio of largest distance over the smallest distance (Gibson *et al.*, SWAT'08) $\rightarrow$ QPTAS for MSR

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Overview of main results

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There is a PTAS for the Euclidean MSD which runs in $n^{O(1/\epsilon)}$

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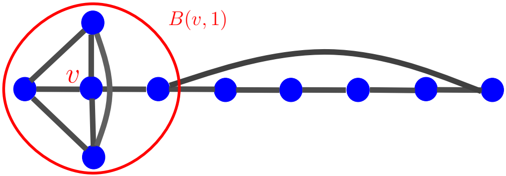

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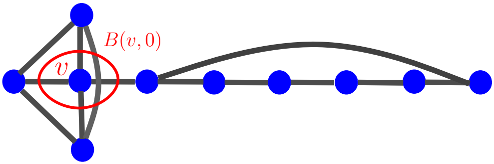

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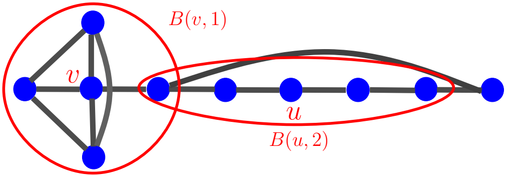

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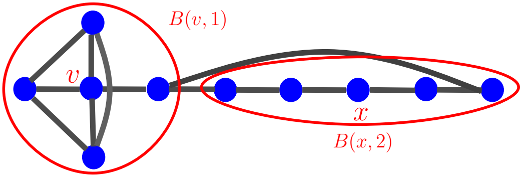

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Canonical optimal solution has minimum number of balls

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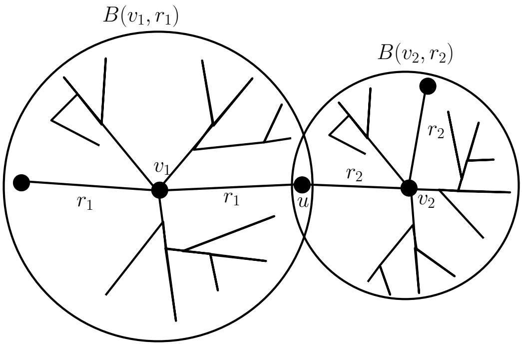
Proof: Substitute these balls with $B(v, r_1 + r_2)$.



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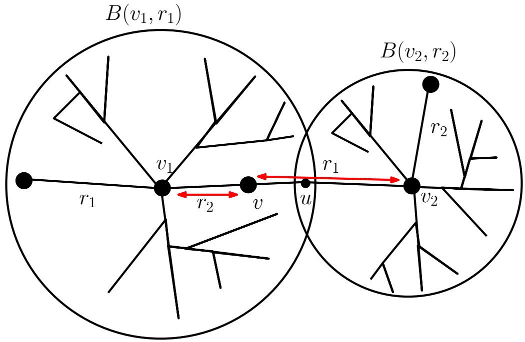
Proof: Choose v at distance r_2 from v_1 on path v_1–v_2.



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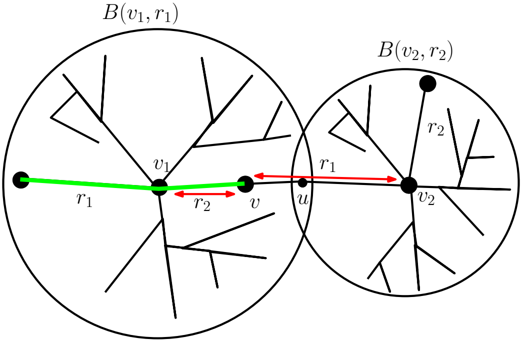

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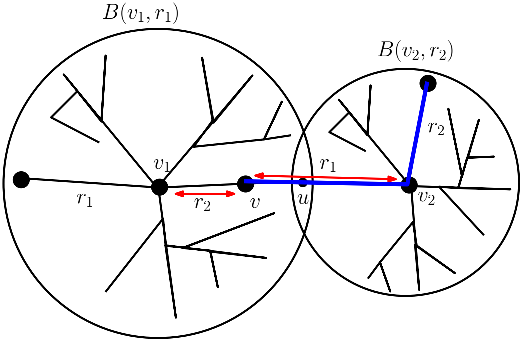

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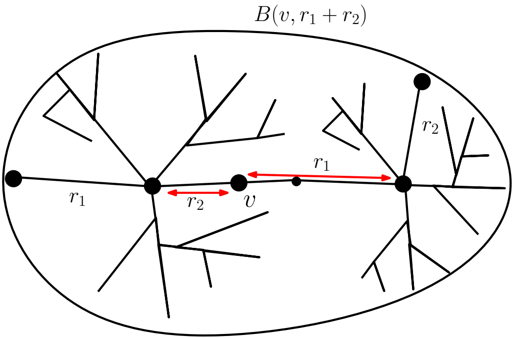

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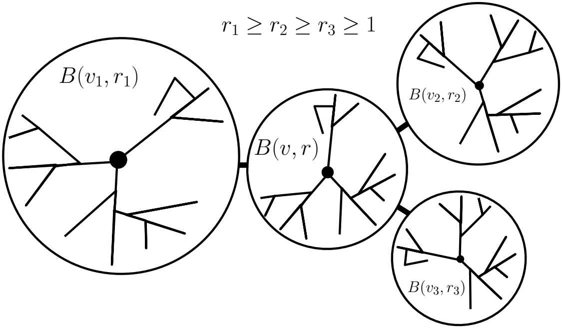

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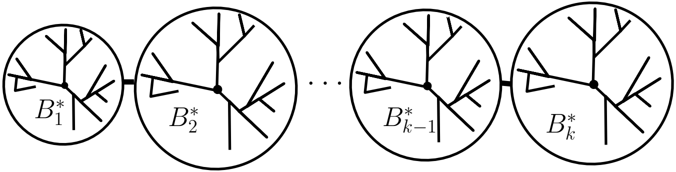
- Consider a canonical optimal solution: $B_1^*, B_2^*, \ldots, B_k^*$.



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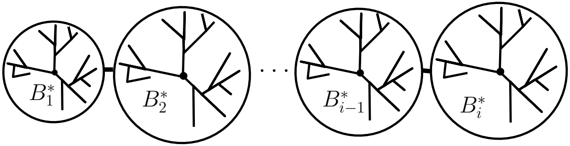

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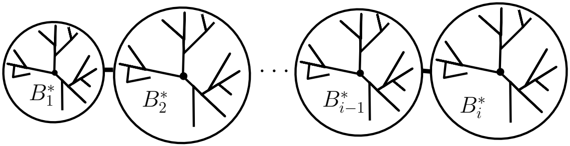

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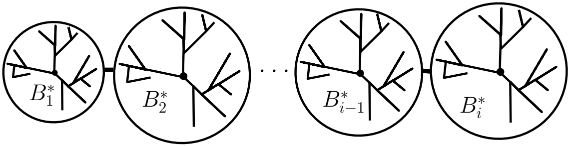

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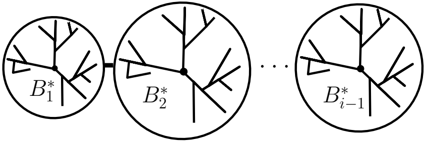

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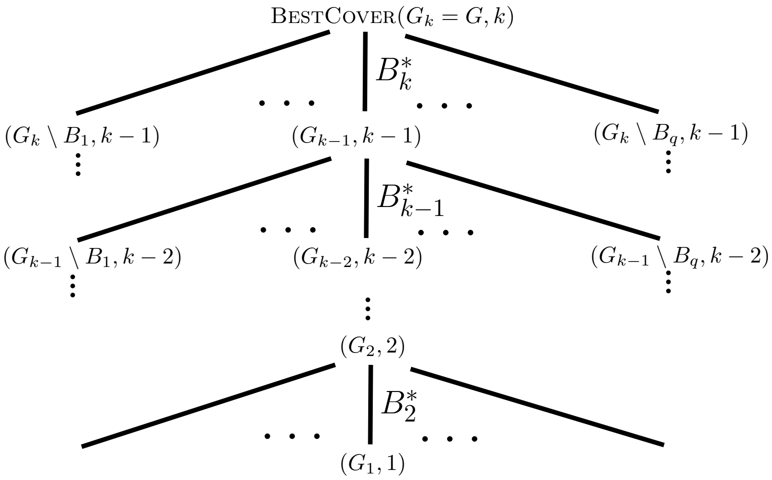

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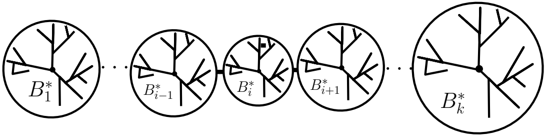

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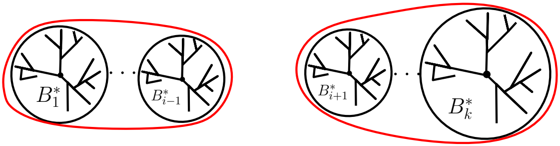

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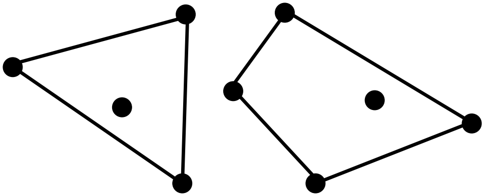

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- Recursively solve each part using dynamic programming.

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- Exponential possible clusters.
- Thin clusters $\rightarrow$ some packing arguments fails.
- Our modifications $\rightarrow$ analysis should be changed.

Handling the first issue

- Approximate each polygon with an enclosing polygon of diameter within factor $(1+\epsilon)$.
- New polygon is simpler: determined by $O(1/\epsilon)$ points $\rightarrow O(n^{1/\epsilon})$ new polygons.
- size c subsets of new polygons, enumerable in $O(n^{\frac{c}{\epsilon}})$.
- Intuitive Example: A regular polygon and the polygon constructed from extension of every ith edges of it

Conclusion and Future Works

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- The MSD problem with constant k: we found an exact algorithm.

Thanks for your attention!
Questions?